

# The Ginsparg-Wilson relation and local chiral random matrix theory

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## Abstract

A chiral random matrix model with locality is constructed and examined. The Nielsen-Ninomiya no-go theorem is circumvented by the use of a generally applicable modified Dirac operator which respects the Ginsparg-Wilson relation. We observe the expected universal behaviour of the eigenvalue density in the microscopic limit.

## Introduction

The success of chiral random matrix models,  $\chi$ RMM, in describing certain aspects of non-perturbative QCD indicates that the correlators in the Dirac spectrum are universal and suggests that the nature of the spontaneous breaking of chiral symmetry does not depend on the detailed dynamical properties of the gauge field. (See Jackson and Verbaarschot [1] and Wettig, Schäfer, and H.A. Weidenmüller [2]. For a comprehensive review, see Verbaarschot [3].) Here, we want to extend these considerations by constructing a local  $\chi$ RMM through the explicit inclusion of the Euclidean derivative terms of the free Dirac operator. There are several reasons for the construction of such models. In practice, the range of validity of random matrix theory in describing real lattice gauge simulations is determined by the ability of the gauge field to mix free quark eigenstates. (In condensed matter physics, this range is known as the “Thouless energy”.) The completely democratic treatment of all basis states in usual chiral random matrix models makes it impossible to address this question. For the same reason, local quantities, e.g., propagators, cannot be considered in usual  $\chi$ RMM. This limitation can be overcome by the construction of a local  $\chi$ RMM.

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In order to represent the Euclidean derivative by a finite size matrix, it is necessary to discretize a finite (Euclidean) space-time volume. This leads inevitably to the problem of unintended fermion doubling. If one insists that chiral symmetry maintain its canonical form,  $\{D, \gamma_5\} = 0$ , on the lattice, this doubling cannot be avoided. In order to obtain a sensible local  $\chi$ RMM, we must resolve the fermion doubling in a manner which preserves a natural extension of chiral symmetry for finite volumes and lattice spacings. The impossibility of constructing a lattice Dirac operator,  $D$ , with locality and exact chiral symmetry but free of fermion doubling is a consequence of the Nielsen-Ninomiya no-go theorem [4]. Some years ago, Ginsparg and Wilson [5] suggested the possibility of circumventing this no-go theorem by modifying the chiral symmetry condition to the form  $\{D, \gamma_5\} = aD\gamma_5D$ . Following the recognition by Hasenfratz [6] that fixed point actions of QCD satisfy the Ginsparg-Wilson relation, attention has again been drawn to lattice theories which satisfy the Ginsparg-Wilson relation, see Hasenfratz, Laliena, and Niedermayer [7], Hasenfratz [8], Neuberger [9, 10], Lüscher [11], and Narayanan [12]. Lüscher [11] has shown that the Ginsparg-Wilson relation ensures an exact “lattice chiral symmetry” of the fermion action for any finite lattice size. Because of this symmetry and the fact that the explicit breaking of chiral symmetry enters through a simple condition on  $D$ , the Ginsparg-Wilson approach suggests itself as a suitable way of avoiding the no-go theorem in a local  $\chi$ RMM. Adopting this approach, it is possible to find a resolution of the fermion doubling problem and embed it in a local  $\chi$ RMM.

The object of the present letter is to present a general procedure for the construction of a Dirac operator which satisfies the Ginsparg-Wilson relation and solves the fermion doubling problem. This procedure is applicable to lattice QCD as well as to the local  $\chi$ RMM which we construct. Further, we shall consider the spectral properties of this local  $\chi$ RMM and demonstrate the universality of the microscopic spectral density, i.e., the spectral density near eigenvalue zero on the scale of the lowest Dirac eigenvalue.

## The Ginsparg-Wilson relation and its implementation

The Ginsparg-Wilson (GW) relation is a constraint on the Dirac operator  $D$

$$D\gamma_5 + \gamma_5D = aD\gamma_5D, \quad (1)$$

where  $a$  is the lattice spacing. The Dirac operator appears as the heart of the fermionic sandwich,  $\bar{\psi}D\psi$ , in the fermionic part of the QCD action

$$\mathcal{S}_F = a^4 \sum_x \bar{\psi}D\psi. \quad (2)$$

Since the anti-commutator of  $D$  and  $\gamma_5$  is not zero, chiral symmetry is explicitly broken. Instead, as shown by Lüscher [11], a new “lattice chiral symmetry” is present. The infinitesimal variation of the fermion fields associated with this new symmetry is

$$\psi \rightarrow \psi + \varepsilon\gamma_5(1 - \frac{1}{2}aD)\psi \quad \text{and} \quad \bar{\psi} \rightarrow \bar{\psi} + \varepsilon\bar{\psi}(1 - \frac{1}{2}aD)\gamma_5. \quad (3)$$

Extension of this singlet flavour lattice chiral symmetry to a non-singlet lattice chiral symmetry is straightforward. Both variations are symmetries of the fermionic action provided that the Dirac operator satisfies the GW relation. Thus, the GW relation ensures the presence of a continuous symmetry which can be regarded as the remnant of chiral symmetry on finite lattices. The Nielsen-Ninomiya no-go theorem states that lattice Dirac operator cannot simultaneously satisfy a series of physically reasonable demands. Among these is that  $\{D, \gamma_5\} = 0$ . By explicitly modifying the form of chiral symmetry through the GW relation to the form of eqn. (3), the no-go theorem can be circumvented. It is necessary to adopt a strategy for dealing with the fermion doubling problem. Here, we choose to work with the Wilson lattice realization of the continuum operator,  $H = \gamma_5[1 - \gamma_\mu(\partial_\mu + iA_\mu)]$ , but this choice is not mandatory. Let us first see how to implement the GW relation.

Consider the Dirac operator,  $D$ , in an arbitrary matrix representation. Since  $D$  is to be multiplied by the row,  $\bar{\psi}$ , from the left and by the column,  $\psi$ , from the right,  $D$  must be a square matrix of dimension, say,  $2N \times 2N$ . For some  $2N \times 2N$  matrix  $\epsilon(H)$ , we can write  $D$  as

$$D = \frac{1}{a}(\mathbf{1} - \gamma_5 \epsilon(H)) , \quad (4)$$

where<sup>1</sup>  $\gamma_5 = \text{diag}(1, 1, \dots, 1, -1, -1, \dots, -1)$ . Since  $\gamma_5$  is invertible, there is a one-to-one correspondence between  $D$  and  $\epsilon(H)$ . As noted by Narayanan [12], the requirement that  $D$  satisfies the GW relation is equivalent to

$$(\epsilon(H))^2 = \mathbf{1} . \quad (5)$$

Since the dynamics of  $D$  enters through  $\epsilon(H)$ , one must build into  $\epsilon(H)$  some regularization of the naïve discretization which effectively eliminates fermion doubling. Having chosen a suitable form for  $H$ , we define the matrix  $\epsilon(H)$  as

$$\epsilon(H) \equiv U_H \text{diag}(\text{sign}(h_1), \text{sign}(h_2), \dots, \text{sign}(h_{2N})) U_H^\dagger \quad (6)$$

where  $U_H$  is the unitary matrix that diagonalizes  $H$ ,

$$U_H^\dagger H U_H = \text{diag}(h_1, h_2, \dots, h_{2N}) . \quad (7)$$

While any choice of  $H$  which provides a suitable regularization of the naïve discretization can be adopted, it is essential for the construction of  $\epsilon(H)$  that  $H$  be hermitian (or antihermitian) and that  $\det(H) \neq 0$ .

Before turning to the construction of a local chiral random matrix model, it is useful to consider some general properties of the spectrum of the resulting Dirac operator and of the order parameter for the finite lattice chiral symmetry. Since  $\epsilon(H)$  is hermitian and satisfies eqn. (5), the combination  $\gamma_5 \epsilon(H)$  is unitary.

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<sup>1</sup>Hasenfratz [7] has shown that the index theorem is reproduced for finite lattice spacing and size provided that the GW relation holds. Thus, the difference in the number of 1 and -1 entries in  $\gamma_5$  depends on the topological index,  $\nu = n_+ - n_-$ , which we shall take as zero.

This implies that  $D$  is normal and that the spectrum of  $D$  lies in the complex plane on the circle,  $(1 - e^{i\theta})/a$  with  $\theta \in [-\pi, \pi]$ . Furthermore,  $D$  satisfies the hermiticity relation,  $\gamma_5 D \gamma_5 = D^\dagger$ . As noted in [13], this property in conjunction with the GW relation implies a complex conjugation symmetry in the spectrum of  $D$ :

$$D\psi_\lambda = \lambda\psi_\lambda \quad (8)$$

$$D\gamma_5\psi_\lambda = \lambda^*\gamma_5\psi_\lambda \quad \text{if } \lambda^* \neq \lambda \quad (9)$$

$$\gamma_5\psi_\lambda = \pm\psi_\lambda \quad \text{if } \lambda^* = \lambda. \quad (10)$$

Using this complex conjugation symmetry, a Banks-Casher-like relation [14] appears in the limits volume  $\rightarrow \infty$  then  $m \rightarrow 0$ , where  $m$  is a regulator mass.

$$\langle \bar{\psi}\psi \rangle = \langle D^{-1}(x, x) \rangle \quad (11)$$

$$= \lim_{m \rightarrow 0} \lim_{\text{Vol} \rightarrow \infty} \lim_{a \rightarrow 0} \frac{1}{\text{Vol}} \int_{-\pi/a}^{\pi/a} \frac{\rho(s)}{(1 - e^{as})/a + m} ds \quad (12)$$

$$= \lim_{m \rightarrow 0} \frac{1}{\text{Vol}} \int_0^\infty \frac{2m\rho(s)}{s^2 + m^2} ds, \quad (13)$$

where the last equality relies on the identity  $\rho(s) = \rho(-s)$ , which is a consequence of the complex conjugation symmetry. The average in eqn. (11) is over gauge field configurations. Finally, we obtain the Banks-Casher relation for the order parameter in the limit of zero lattice spacing

$$\langle \bar{\psi}\psi \rangle = \frac{\pi\rho(0)}{\text{Vol}}. \quad (14)$$

It is also straightforward to evaluate the integral in eqn. (12) in the limit  $m \rightarrow 0$  for any finite  $a$ . One obtains

$$\frac{a}{\text{Vol}} \int_0^{\frac{\pi}{a}} \rho(s) ds + \frac{\pi\rho(0)}{\text{Vol}}. \quad (15)$$

Hasenfratz [8] has suggested the “subtracted” chiral condensate as a possible order parameter for lattice chiral symmetry,

$$\langle \bar{\psi}\psi \rangle_{\text{sub}} = \frac{1}{\text{Vol}} \langle \text{Tr}(D^{-1} - \frac{1}{2}a\mathbf{1}) \rangle, \quad (16)$$

where the trace extends over colour, flavour, and Dirac space indices. By virtue of the GW relation, only zero modes of  $D$  contribute to  $\langle \bar{\psi}\psi \rangle_{\text{sub}}$ . The result of eqn. (15) is that a Banks-Casher relation remains valid at all finite lattice spacings provided one uses the subtracted chiral condensate as the order parameter for the lattice chiral symmetry. Of course, in the limits of infinite volume and  $a \rightarrow 0$ , the lattice chiral symmetry becomes genuine chiral symmetry, and the subtracted chiral condensate evolves into the true chiral condensate.

## A local chiral random matrix model

We shall construct a local  $\chi$ RMM using the standard Wilson prescription for dealing with the fermion doubling problem. We choose as a basis the states obtained by chiral projection from the eigenstates of the naïve lattice realization of the free Euclidian continuum operator  $\gamma_\mu \partial_\mu$ . In this basis,  $H$  is defined as

$$H \equiv \gamma_5 X, \text{ where } X \equiv \mathbf{1} - a \left[ \begin{pmatrix} \Delta' & i\Delta \\ i\Delta & \Delta' \end{pmatrix} + \begin{pmatrix} 0 & iW \\ iW^\dagger & 0 \end{pmatrix} \right]. \quad (17)$$

In general, the matrix  $W$  is uniquely determined by the gauge field configuration. The local  $\chi$ RMM Dirac operator is defined through eqns. (4) and (6) by replacing  $W$  by a suitable random matrix. The  $N \times N$  matrices  $\Delta$  and  $\Delta'$  are real and diagonal with

$$\Delta = \frac{1}{a} \text{diag}(\dots, [\sum_{i=1}^3 \sin^2(ap_i^{(j)}) + \sin^2(a\pi T)]^{\frac{1}{2}}, \dots) \quad (18)$$

$$\Delta' = \frac{1}{a} \text{diag}(\dots, \sum_{i=1}^3 (1 - \cos(ap_i^{(j)})) + 1 - \cos(a\pi T), \dots), \quad (19)$$

where  $-n/2 \leq j \leq n/2 - 1$  with  $n \equiv N^{1/3}$ . The spatial momenta  $p_i^{(j)} = 2\pi j/(an)$  are determined by imposing periodic boundary conditions in the spatial directions. Due to the anti-periodic boundary conditions in the time direction, the temporal momenta are given by the Matsubara frequencies  $(2k+1)\pi T$ , where  $k$  is an integer. As is customary in chiral random matrix models, we have retained only the lowest Matsubara frequency,  $\pi T$ .

It is conventional in  $\chi$ RMM to replace both the known form of the free Dirac operator as well as the gluon field contributions in the random matrix  $W$ . The new feature of the present model is that the free Dirac operator is explicitly retained and  $W$  is assumed to describe only gluonic contributions. We choose the  $N \times N$  matrix  $W$  to be a random complex matrix, and its entries are drawn at random on a Gaussian distribution. This choice of  $W$  is motivated by the conjecture [15] that the correlators of QCD with three colours are given by the chiral Gaussian unitary ensemble. (The Gaussian orthogonal and symplectic ensembles are expected to be relevant for the description of QCD with a smaller number of colours.) Note that the form of the  $W$ -dependent part of  $X$  in eqn. (17) provides the most general antihermitian matrix which preserves the relation,  $\gamma_5 X \gamma_5 = X^\dagger$ , and thus ensures the hermiticity of  $H$ . The only “dynamical” information in  $W$  is its variance. We shall consider this point below.

To see that this implementation of the GW relation yields a physically sensible non-doubled spectrum for the Dirac operator, it is useful to look at the free spectrum. Turning off the gluon field, i.e.,  $W = 0$ , the eigenvalues of the free Dirac operator can be found analytically. Diagonalizing  $D$ , one finds that the eigenvalues,  $\lambda$ , of  $D$  are in one-to-one correspondence with the eigenvalues,

$\xi_{j\pm} = 1 - a\Delta'_{jj} - ai\Delta_{jj}$ , of the free Wilson operator  $X$ . The result is simply  $\lambda = (1 - \xi/|\xi|)/a$ . Evidently, only the phase information of  $\xi$  is carried through to  $\lambda$ . This information, however, is all that is needed in order to resolve the fermion doubling problem: For small momenta, the Wilson term makes a negligible contribution to the real part of  $\xi$ . The corresponding  $\lambda$  eigenvalues appear at small angles,  $\theta$ , on the circle  $(1 - e^{i\theta})/a$ . As the momentum increases, the Wilson term becomes increasingly important with the effect that the doubled states are pushed towards  $\theta = \pm\pi$ . Since the doubled states are thus well separated from the small  $\theta$  region of physical interest, this represents a sensible resolution of the doubling problem expected to remain valid when  $W$  is reintroduced. When  $W$  is not equal to zero,  $X$  is no longer normal and such analytic evaluation is not possible to our knowledge. (Of course,  $H = \gamma_5 X$  is hermitian for both interacting and non-interacting fermions.)

In general the analytical relation between the eigenvalues of  $D$  and the eigenvalues of  $X$ ,  $\lambda = (1 - \xi/|\xi|)/a$ , holds if  $X$  is normal and satisfies  $\gamma_5 X \gamma_5 = X^\dagger$ . Thus, this relation also applies when there is no deterministic part in  $X$ , i.e.,  $\Delta = \Delta' = 0$ . The eigenvalues of  $D$  can then be expressed in terms of the eigenvalues,  $i\omega_j$ , of the matrix

$$\begin{pmatrix} 0 & iW \\ iW^\dagger & 0 \end{pmatrix} \quad (20)$$

which has the usual form of chiral random matrices. The eigenvalues  $i\omega_j$  are simply mapped onto the circle  $(1 - e^{i\theta_j})/a$  according to  $\cos\theta_j = 1/\sqrt{1 + \omega_j}$ . This is sufficient to ensure that both the microscopic spectral density and the various spectral correlators of  $D$  are identical to those of the original  $\chi$ RMM.

## Applications and numerical results

The simplest application of the random matrix model constructed in the preceding section is the study of the chiral condensate as a function of temperature. As indicated by eqn.(16), this requires us to consider the microscopic spectral density of  $D$ . We wish to consider the case where chiral symmetry is spontaneously broken at  $T = 0$ . In ordinary random matrix models, the scale of the problem is set solely by the variance of the random matrix  $W$ . Chiral symmetry is always spontaneously broken at  $T = 0$ , and the critical temperature for its restoration scales strictly with the variance of  $W$ . The situation is somewhat more complicated in the present case where  $D$  contains both deterministic and random elements. Here, the existence of a chiral condensate even at zero temperature is determined by a competition between low-energy free eigenvalues and the variance of the random matrix  $W$ . The spectral properties of matrices which are the sum of deterministic and random parts have been considered in refs. [2, 16, 17] and [18]. This work shows that the resulting spectral correlators and the microscopic spectral density will be given exactly by the strict random matrix results provided only that the random part of the matrix is sufficiently strong. We expect to find a similar result in the present case.

The problem considered explicitly in [2] was a matrix of the form

$$\begin{pmatrix} 0 & \Delta + W \\ \Delta + W^\dagger & 0 \end{pmatrix} \quad (21)$$

where  $\Delta$  is a fixed diagonal matrix and the elements of  $W$  are drawn at random on the Gaussian weight

$$P(W) \sim \exp\{-N\Sigma^2 \text{Tr}WW^\dagger\} . \quad (22)$$

The parameter  $\Sigma$  introduced here determines the variance of the distribution. When the diagonal elements of  $\Delta$  are all non-zero, the chiral condensate,  $\Xi$ , can be expressed as

$$\Xi = \Sigma^2 \bar{x} , \quad (23)$$

where  $\bar{x}$  is the only real and positive solution of

$$N\Sigma^2 = \sum_j \frac{1}{\Delta_{jj}^2 + \bar{x}^2} . \quad (24)$$

In the absence of a solution to eqn. (24), the chiral condensate is zero. Noting that the right side of eqn. (24) is a monotonically decreasing function of  $\bar{x}$ , it follows that a necessary condition for a non-zero chiral condensate in the presence of the deterministic matrix  $\Delta$  is

$$N\Sigma^2 \leq \sum_j \frac{1}{\Delta_{jj}^2} . \quad (25)$$

Of course, neither the matrices  $X$ ,  $H$ , or  $D$  of the present model have the form of eqn. (21). Nevertheless, we would like to use eqns. (24) and (25) to provide a rough estimate of the variance of  $W$  required to ensure a reasonable critical temperature for chiral symmetry restoration. These equations are naturally dominated by the lowest eigenvalues of  $\Delta$ . Turning to the matrix  $X$  defined by eqns. (17)–(19), we see that these lowest states correspond to the “undoubled” fermion states. Restricting our attention only to these states, it is reasonable to neglect the Wilson term,  $\Delta'$ . This leaves us with a matrix having the form of eqn. (21).

Having first determined the size,  $N$ , of the matrix and the lattice spacing,  $a$ , we pick the variance of  $W$  in order to fix the estimated critical temperature at some value,  $T_c$ . Thus,

$$N\Sigma^2 = \sum_{j=1}^{N/8} \frac{1}{(\Delta_{jj}(T_c))^2} . \quad (26)$$

Numerical simulations indicate that this procedure is sound in practice and that the resulting critical temperature is close to this estimated value.

As an initial application of this local  $\chi$ RMM, we have performed a numerical study of the microscopic level density. As indicated above, we anticipate that

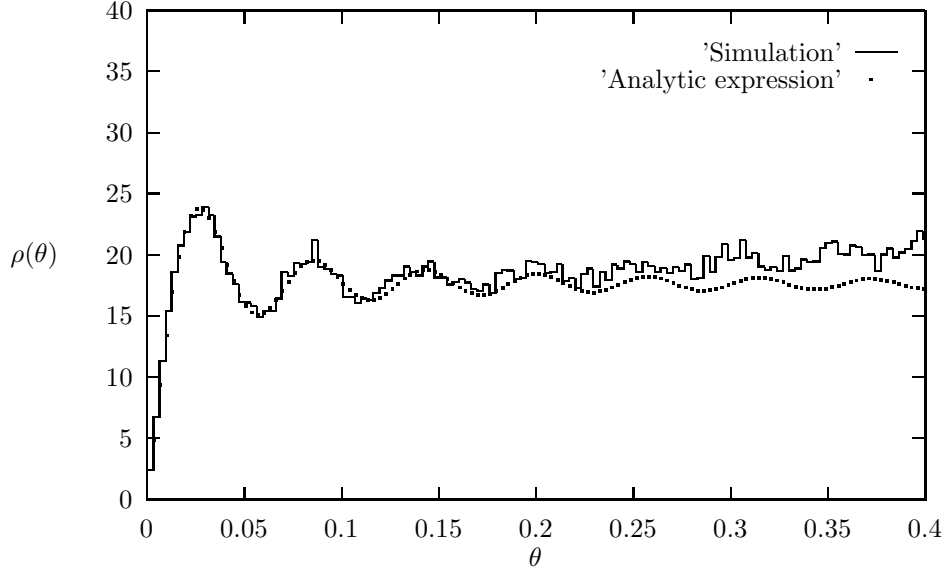


Figure 1: Universal behaviour of the eigenvalue density for the Dirac operator of the local  $\chi$ RMM. The histogram shows the microscopic level density at zero temperature while the smooth dotted curve is the analytic Bessel function expression suitably scaled. This simulation was performed for a  $24 \times 24$  lattice with a lattice spacing of  $1/4$  fm and a critical temperature of approximately 150 MeV.

the microscopic spectral density for our model will be identical to the usual  $\chi$ RMM result, which can be expressed in terms of the first two Bessel functions [19] and [20] as:

$$\rho(\lambda) = 2N\Xi(N\Xi\lambda)[J_0^2(2N\lambda\Xi) + J_1^2(2N\lambda\Xi)] . \quad (27)$$

The analogous result for the Gaussian symplectic ensemble has been shown to be in excellent agreement with the results of QCD lattice simulations at low temperature. Eqn. (27) applies whenever chiral symmetry is spontaneously broken, and the magnitude of the chiral condensate is directly related to the asymptotic value of the microscopic spectral density. As anticipated, numerical results obtained for a variety of temperatures below  $T_c$  are in agreement with eqn. (27). Results for the  $\rho$  at  $T = 0$  are shown in the figure. The density is plotted as a function of  $\theta$  with  $\lambda = (1 - \exp[i\theta])/a$ . The disagreement between analytic and numerical results seen for larger values of  $\theta$  is a consequence of both finite size effects and, more importantly, the limited ability of the random matrix,  $W$ , to mix free quark states of very different momenta.

## Conclusions and discussion

We have suggested a simple and well-defined procedure for modifying the usual Wilson Dirac operator in order to satisfy the Ginsparg-Wilson relation. The result is a lattice Dirac operator which both solves the fermion doubling problem and retains a lattice chiral symmetry for all lattice spacings and sizes. As noted, this lattice chiral symmetry becomes genuine chiral symmetry in the continuum limit. While we have used this procedure to construct a local chiral random matrix model, we emphasise that it is more generally applicable. (For example, the procedure outlined here would be suitable for QCD lattice simulations.) As is common in  $\chi$ RMM, we have introduced temperature dependence through the lowest Matsubara frequency. We have studied the microscopic level density of the resulting Dirac operator numerically and confirmed the anticipated agreement with the universal behaviour found empirically in QCD lattice data and analytically in other chiral random matrix models. This allows us to extract the continuum limit (i.e.,  $N \rightarrow \infty$ ) of the lattice chiral condensate from finite-size simulations. The existence of this lattice chiral symmetry for every  $a$  leads us to anticipate that the limit  $a \rightarrow 0$ , in which lattice chiral symmetry becomes genuine chiral symmetry, will be relatively smooth.

This local  $\chi$ RMM represents an appealing intermediate step between pure random matrix models and real QCD lattice simulations. There appear to be a variety of interesting applications of local  $\chi$ RMM. The presence of the free Dirac operator (in a form free of fermion doubling problems) makes it sensible to consider a number of local properties which are genuinely not accessible to ordinary random matrix theory. As noted, it is possible to explore the interplay between random and deterministic elements in this model and to study the range of applicability of random matrix techniques to lattice simulations. It is now possible to search for quasi-universal behaviour in correlators and to address questions regarding localisation. While such studies will initially be numerical, they are far simpler than full QCD lattice simulations and may well have interesting insights to offer.

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